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3-State Potts Model and Automorphisms of Vertex Operator Algebras of Order 3

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We study the fusion rules of a vertex operator algebra $W(0)^+_{\mathbb{R}}$, which is a VOA over the real number field $\mathbb{R}$ and has a positive definite invariant bilinear form, and such that its complexification $\mathbb{C}W(0)^+_{\mathbb{R}}$ is a direct sum of the 3-state Potts model $L(\frac{4}{5},0)$ and its module $L(\frac{4}{5},3)$. As an application, we define an automorphism of order 3 of a VOA $V_{\mathbb{R}}$ over $\mathbb{R}$ if $V_{\mathbb{R}}$ contains $W(0)^+_{\mathbb{R}}$ as a sub VOA.

1. INTRODUCTION

For finite group theorists, the most interesting vertex operator algebra (VOA) is the moonshine vertex operator $V^{\natural}_{\mathbb{R}}$ over the real number field $\mathbb{R}$, whose second primary part $V^{\natural}_2$ is the Griess algebra B and the full automorphism group is the monster simple group $\mathbb{M}$ [FrLeMe]. Our main purposes are to study a VOA $V_{\mathbb{R}}$ over $\mathbb{R}$ and to define an automorphism of a VOA $V_{\mathbb{R}}$. Throughout this paper, $V_{\mathbb{R}}$ denotes a VOA over $\mathbb{R}$ and $\mathbb{C}V_{\mathbb{R}}$ denotes its complexification $\mathbb{C}\otimes_{\mathbb{R}} V$. Namely, the subscript $*_{\mathbb{R}}$ denotes that a VOA (or a module) is over $\mathbb{R}$. For example, $L(c,0)_{\mathbb{R}}$ denotes a simple vertex operator algebra over $\mathbb{R}$ generated by a Virasoro element of central charge $c\in\mathbb{R}$ and $\mathbb{C}L(c,0)_{\mathbb{R}}$ denotes its complexification, which is equal to an ordinary simple Virasoro VOA $L(c,0)$ over $\mathbb{C}$ with central charge c as we will show in Section 2. At first, we will show in Section 2 that there is no essential difference between the representation theory of $V_{\mathbb{R}}$ over $\mathbb{R}$ and that of $\mathbb{C}V_{\mathbb{R}}$ over $\mathbb{C}$. For example, we will prove that if $\mathbb{C}V_{\mathbb{R}}$ is rational, then so is $V_{\mathbb{R}}$, where a rational VOA is a VOA which has only finitely many irreducible modules and every module is completely reducible.

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In research about the Griess algebra $B_{\mathbb{R}}$ over $\mathbb{R}$, Conway [C] found several idempotents, called "axes," of the Griess algebra corresponding to elements of $\mathbb{M}$.

It was proved in [Mi] that an idempotent in the Griess algebra is half of a conformal vector (or a Virasoro element of a sub VOA). In particular, every idempotent in Conway's list is half of the Virasoro element of a sub VOA which is isomorphic to one of the minimal discrete series of Virasoro VOA $L(c,0)_{\mathbb{R}}$. We note that a sub VOA $(W = \oplus_{n\in\mathbb{Z}} W_n, Y', \mathbf{1}, e)$ of $(V = \oplus_{n\in\mathbb{Z}} V_n, y, \mathbf{1}, \omega)$ in this paper does not usually contain the Virasoro element ω of V, but the grading on W given by $L'(0)$ is the same as the grading given by $L(0)$; that is, $W_n = \{v \in W \mid L'(0)v = nv\} = \{v \in W \mid L(0)v = nv\} = W \cap V_n$, where $Y(\omega, z) = \sum L(n)z^{-n-2}$ and $Y'(e,z) = \sum L'(n)z^{-n-2}$; see Section 7 for more detail. For example, an idempotent corresponding to a 2A-involution of the Monster simple group is half of the Virasoro element of a sub VOA isomorphic to the two-dimensional Ising model $L(\frac{1}{2},0)_{\mathbb{R}}$ with central charge $\frac{1}{2}$. Conversely, the author showed in [Mi] that if a VOA V contains a sub VOA W isomorphic to the Ising model $L(\frac{1}{2},0)$, then it defines an automorphism τ_W of V of order at most 2. The arguments in this paper will easily show that this result is also true for a VOA $V_{\mathbb{R}}$ over $\mathbb{R}$ containing $L(\frac{1}{2},0)_{\mathbb{R}}$.

In Conway's list, an idempotent for a 3A element is half of the Virasoro element of a sub VOA isomorphic to $L(\frac{4}{5},0)_{\mathbb{R}}$ with central charge $\frac{4}{5}$ and its complexification $\mathbb{C}L(\frac{4}{5},0)_{\mathbb{R}}$ is equal to the 3-state Potts model $L(\frac{4}{5},0)$. So it is natural for us to expect an automorphism g (of order 3) defined by a sub VOA isomorphic to $L(\frac{4}{5},0)_{\mathbb{R}}$. Since we want to define an automorphism of order 3 using eigenvalues, we have to consider a VOA over $\mathbb{C}$. It is known that $L(\frac{4}{5},0)$ is a rational VOA and has the irreducible modules (cf. [W])

$$L\left(\tfrac{4}{5},0\right), L\left(\tfrac{4}{5},\tfrac{1}{8}\right), L\left(\tfrac{4}{5},\tfrac{2}{3}\right), L\left(\tfrac{4}{5},\tfrac{13}{8}\right), L\left(\tfrac{4}{5},3\right),$$
$$L\left(\tfrac{4}{5},\tfrac{2}{5}\right), L\left(\tfrac{4}{5},\tfrac{1}{40}\right), L\left(\tfrac{4}{5},\tfrac{1}{15}\right), L\left(\tfrac{4}{5},\tfrac{21}{40}\right), L\left(\tfrac{4}{5},\tfrac{7}{5}\right),$$

where the second entries are highest weights.

It is clear that it is not enough to think of only $L(\frac{4}{5},0)$ in order to define an automorphism g of $V_{\mathbb{R}}$ of order 3, because if V^1 and V^2 are the eigenspaces of g in $\mathbb{C}V_{\mathbb{R}}$ with eigenvalues $e^{2\pi i/3}$ and $e^{4\pi i/3}$, respectively, then they are isomorphic as $L(\frac{4}{5},0)$-modules. However, Dong and Mason [DMa] showed a wonderful result that V^1 and V^2 are not isomorphic as $\mathbb{C}V_{\mathbb{R}}^{\langle g\rangle}$-modules, where $V_{\mathbb{R}}^{\langle g\rangle}$ is the space of g-invariants of $V_{\mathbb{R}}$. So we have to think of a larger sub VOA. What is the difference between $\mathbb{C}V_{\mathbb{R}}^{\langle g\rangle}$ and $L(\frac{4}{5},0)$? Recently, Kitazume et al. constructed a new class of VOAs over the complex field by using codes over $\mathbb{Z}_3$ in [KMiY]. The interesting point is that they used a VOA isomorphic to $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ as a sub VOA corresponding to $0 \in \mathbb{Z}_3$. As a simple VOA, $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ has a unique VOA structure $W(0)$ as we will see, but $L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}$ has two VOA structures over $\mathbb{R}$. One of them has a positive definite invariant bilinear form, which is the desired VOA and denoted by $W(0)^+_{\mathbb{R}}$. It is known in a physics literature [FaZa] that the $W(0)$-modules have a $\mathbb{Z}_3$-symmetry. We will prove this result and the following theorem:

THEOREM 5.1. *If a VOA V over $\mathbb{C}$ contains a sub VOA X isomorphic to $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$, then an endomorphism σ_X of V defined by*

$$\sigma_X : \begin{cases} 1 & \text{on } W(0) \quad \text{and} \quad W\left(\frac{2}{5}\right) \\ e^{2\pi i/3} & \text{on } W\left(\frac{2}{3},+\right) \quad \text{and} \quad W\left(\frac{1}{15},+\right) \\ e^{4\pi i/3} & \text{on } W\left(\frac{2}{3},-\right) \quad \text{and} \quad W\left(\frac{1}{15},-\right) \end{cases}$$

is an automorphism of V.

We will next classify all irreducible $W(0)^+_{\mathbb{R}}$-modules.

THEOREM 6.1. *$\mathbb{C}W(0)^+_{\mathbb{R}}$ is isomorphic to $W(0)$ as a VOA over $\mathbb{C}$. $W(0)^+_{\mathbb{R}}$ is a rational VOA and it has exactly the following four irreducible modules*:

$$W(0)^+_{\mathbb{R}}, W\left(\frac{2}{5}\right)^+_{\mathbb{R}}, W\left(\frac{2}{3}\right)^+_{\mathbb{R}}, W\left(\frac{1}{15}\right)^+_{\mathbb{R}}.$$

Using $W(0)^+_{\mathbb{R}}$, we will define an automorphism of $V_{\mathbb{R}}$.

THEOREM 6.2. *Assume that a VOA $V_{\mathbb{R}}$ over $\mathbb{R}$ contains a sub VOA $W_{\mathbb{R}}$ isomorphic to $W(0)^+_{\mathbb{R}}$. Then the endomorphism $\sigma_{\mathbb{C}W_{\mathbb{R}}}$ of $\mathbb{C}V_{\mathbb{R}}$ defined by Theorem* 5.1 *keeps $V_{\mathbb{R}}$ invariant. In particular, $(\sigma_{\mathbb{C}W_{\mathbb{R}}})_{|V_{\mathbb{R}}}$ is an automorphism of $V_{\mathbb{R}}$.*

In this paper, we often view V as a W-module (or an infinite direct sum of W-modules) if V contains a sub VOA W. This is not obvious since one of the axioms of VOA-modules expects the grade preserving operator e_1 of the Virasoro element e of W to act on V diagonally, where $Y(e,z) = \sum_{i\in\mathbb{Z}} e_i z^{-i-1}$ is the vertex operator of e. In Section 6, we will prove that this is generally true for a rational sub VOA W.

2. PRELIMINARY RESULTS

In this paper, V (or $V_{\mathbb{R}}$) is a VOA and $Y(v,z) = \sum_{i\in\mathbb{Z}} v_i z^{-i-1}$ denotes the vertex operator of v. By abuse of notation, we also use it for vertex operators for modules. At first, we will prove some basic results for a VOA over $\mathbb{R}$.

THEOREM 2.1. *Let $V_{\mathbb{R}}$ be a VOA over $\mathbb{R}$ and let U be an irreducible $\mathbb{C}V_{\mathbb{R}}$-module with real weights; then U is an irreducible $V_{\mathbb{R}}$-module or there is a unique $V_{\mathbb{R}}$-module $U_{\mathbb{R}}$ such that $\mathbb{C}U_{\mathbb{R}} \cong U$ as $\mathbb{C}V_{\mathbb{R}}$-modules.*

Proof. Let ω be the Virasoro element of $V_{\mathbb{R}}$. It is clear that $L(0) := \omega_1$ acts diagonally on U with real eigenvalues and so U is a module of $V_{\mathbb{R}}$. Assume that U is not an irreducible $V_{\mathbb{R}}$-module and $\tilde{U}_{\mathbb{R}}$ is a proper $V_{\mathbb{R}}$-submodule. Then $\tilde{U}_{\mathbb{R}} \cap \sqrt{-1}\,\tilde{U}_{\mathbb{R}}$ is a $\mathbb{C}V_{\mathbb{R}}$-submodule and so $\tilde{U}_{\mathbb{R}} \cap \sqrt{-1}\,\tilde{U}_{\mathbb{R}} = 0$. Hence we have $\mathbb{C}\tilde{U}_{\mathbb{R}} = U$ and $U \cong \tilde{U}_{\mathbb{R}} \oplus \tilde{U}_{\mathbb{R}}$ as $V_{\mathbb{R}}$-modules. In particular, any proper $V_{\mathbb{R}}$-submodule of U is isomorphic to $\tilde{U}_{\mathbb{R}}$. ∎

We should note that the eigenvalues of $L(0)$ on $V_{\mathbb{R}}$-modules should be real. From now on, we assume $c \in \mathbb{R}$ and $L(c,0)_{\mathbb{R}}$ denotes a simple vertex operator algebra over $\mathbb{R}$ generated by a Virasoro element of central charge $c \in \mathbb{R}$ and $\mathbb{C}L(c,0)_{\mathbb{R}}$ denotes its complexification.

LEMMA 2.1. *$\mathbb{C}L(c,0)_{\mathbb{R}}$ is isomorphic to $L(c,0)$ as a VOA.*

Proof. Clearly, $L(c,0)$ is an irreducible $\mathbb{C}L(c,0)_{\mathbb{R}}$-module. Since $\dim_{\mathbb{R}}(L(c,0)_0) = 2$, $L(c,0)$ is not an irreducible $L(c,0)_{\mathbb{R}}$-module and so there is an irreducible $L(c,0)_{\mathbb{R}}$-module $W_{\mathbb{R}}$ with highest weight 0 such that $L(c,0) = \mathbb{C}W_{\mathbb{R}}$ by Theorem 2.1. Clearly, $W_{\mathbb{R}} \cong L(c,0)_{\mathbb{R}}$ and so $L(c,0) \cong \mathbb{C}L(c,0)_{\mathbb{R}}$. ∎

COROLLARY 2.1. *Assume that $L(c,h)$ is an irreducible $L(c,0)$-module with highest weight h. If $h \in \mathbb{R}$, then there is a unique irreducible $L(c,0)_{\mathbb{R}}$-module $L(c,h)_{\mathbb{R}}$ such that $L(c,h) \cong \mathbb{C}L(c,h)_{\mathbb{R}}$. If $h \notin \mathbb{R}$, then $L(c,h)$ is an irreducible $Vir_{\mathbb{R}}$-module, where $Vir_{\mathbb{R}}$ is a Virasoro algebra over $\mathbb{R}$.*

Proof. Assume $h \in \mathbb{R}$ and let $v \in L(c,h)_h$ be a highest weight vector. Then $L(c,h)$ contains a highest weight $L(c,h)_{\mathbb{R}}$-submodule W with highest weight vector v. Since $\dim_{\mathbb{R}}(L(c,h)_h) = 2$, $L(c,h)$ is not an irreducible $Vir_{\mathbb{R}}$-module and so we have the desired conclusion by Theorem 2.1. Assume $h \notin \mathbb{R}$ and $L(c,h)$ is not irreducible as a $Vir_{\mathbb{R}}$-module. Let $W_{\mathbb{R}}$ be a proper $Vir_{\mathbb{R}}$-submodule of $L(c,h)$. Set $U_{\mathbb{R}} = \sum_{n_1,\dots,n_k<0} \mathbb{R}L(n_1)\cdots L(n_k)v$. It is easy to see $L(c,h) = U_{\mathbb{R}} + hU_{\mathbb{R}}$ since $h \notin \mathbb{R}$. If $W_{\mathbb{R}}$ contains v, then $W_{\mathbb{R}} = L(c,h)$ since $L(0)v = hv$, which is a contradiction. Hence the lowest degree of $W_{\mathbb{R}}$ is greater than h, which means that $L(c,h)$ contains another highest weight vector and so we have a contradiction. ∎

We will next classify all irreducible $L(c,0)_{\mathbb{R}}$-modules.

LEMMA 2.2. *If $W_{\mathbb{R}}$ is an irreducible $L(c,0)_{\mathbb{R}}$-module, then there is $h \in \mathbb{R}$ such that $W \cong L(c,h)_{\mathbb{R}}$ and $L(c,h)$ is an irreducible $L(c,0)$-module.*

Proof. It is clear that if $W_{\mathbb{R}}$ is an irreducible $L(c,0)_{\mathbb{R}}$-module, then it is a highest weight $< L(m) : m \in \mathbb{Z} >_{\mathbb{R}}$-module with highest weight $h \in \mathbb{R}$.

In particular, $\dim(W_{\mathbb{R}})_h = 1$. $\mathbb{C}W_{\mathbb{R}}$ is also a highest weight $L(c,0)$-module with highest weight h. If $\mathbb{C}W_{\mathbb{R}}$ is irreducible, then $\mathbb{C}W_{\mathbb{R}} = L(c,h)$ and so $W \cong L(c,h)_{\mathbb{R}}$ by the definition of $L(c,h)_{\mathbb{R}}$. So we assume that $\mathbb{C}W_{\mathbb{R}}$ is not irreducible; then it contains a maximal $L(c,0)$-submodule U such that $\mathbb{C}W_{\mathbb{R}}/U \cong L(c,h)$. In particular, $U_h = 0$. Since $\mathbb{C}W_{\mathbb{R}} \cong W_{\mathbb{R}} \oplus W_{\mathbb{R}}$ as $L(c,0)_{\mathbb{R}}$-modules, we have $L(c,h) \cong U \cong L(c,h)_{\mathbb{R}}$ as $L(c,0)_{\mathbb{R}}$, which contradicts $U_h = 0$. ∎

Therefore, the set of irreducible $L(c,0)_{\mathbb{R}}$-modules is

$$\{L(c,h)_{\mathbb{R}} \mid h \in \mathbb{R}, L(c,h) \text{ is an irreducible } L(c,0)\text{-module}\}.$$

In particular, if $L(c,0)$ is one of minimal discrete series, then all highest weights are real numbers and so there is one to one correspondence between irreducible $L(c,0)$-modules and irreducible $L(c,0)_{\mathbb{R}}$-modules.

THEOREM 2.2. *If $\mathbb{C}V_{\mathbb{R}}$ is rational, then so is $V_{\mathbb{R}}$.*

Proof. Since the number of isomorphism classes of irreducible $\mathbb{C}V_{\mathbb{R}}$-components in $\mathbb{C}W_{\mathbb{R}}$ is at most 2 for an irreducible $V_{\mathbb{R}}$-module $W_{\mathbb{R}}$ and the number of isomorphism classes of irreducible $V_{\mathbb{R}}$-submodules in an irreducible $\mathbb{C}V_{\mathbb{R}}$-module is also at most 2, the number of isomorphism classes of irreducible $V_{\mathbb{R}}$-modules is finite. We next show that all modules are completely reducible. Suppose this is false and let $U_{\mathbb{R}}$ be a minimal counterexample. In particular, a maximal proper $V_{\mathbb{R}}$-submodule $W_{\mathbb{R}}$ of $U_{\mathbb{R}}$ is a completely reducible. We will show that $W_{\mathbb{R}}$ is irreducible. Suppose that this is false and $W_{\mathbb{R}}$ contains a proper submodule $W^1_{\mathbb{R}}$. Then $U_{\mathbb{R}}/W^1_{\mathbb{R}}$ is completely reducible and so $U_{\mathbb{R}}/W^1_{\mathbb{R}} = U^1_{\mathbb{R}}/W^1_{\mathbb{R}} \oplus W_{\mathbb{R}}/W^1_{\mathbb{R}}$ as $V_{\mathbb{R}}$-modules. By the minimality of $U_{\mathbb{R}}$, $U^1_{\mathbb{R}}$ has a complement of $W^1_{\mathbb{R}}$, which is also a complement of $W_{\mathbb{R}}$ in $U_{\mathbb{R}}$, a contradiction. So $W_{\mathbb{R}}$ and $U_{\mathbb{R}}/W_{\mathbb{R}}$ are irreducible and a vertex operator $Y^{U_{\mathbb{R}}}(v,z)$ for $U_{\mathbb{R}}$ has the form

$$Y^{U_{\mathbb{R}}}(v,z) = \begin{pmatrix} Y^1(v,z) & Y^2(v,z) \\ 0 & Y^3(v,z) \end{pmatrix}$$

for any $v \in V_{\mathbb{R}}$, where $Y^1(v,z) \in \mathrm{End}(W_{\mathbb{R}})[[z,z^{-1}]]$, $Y^2(v,z) \in \mathrm{Hom}(U_{\mathbb{R}}/W_{\mathbb{R}}, W_{\mathbb{R}})[[z,z^{-1}]]$, and $Y^3(v,z) \in \mathrm{End}(U_{\mathbb{R}}/W_{\mathbb{R}})[[z,z^{-1}]]$. By the assumption, $\mathbb{C}U_{\mathbb{R}}$ is a completely reducible and so $\mathbb{C}U_{\mathbb{R}} = \mathbb{C}W_{\mathbb{R}} \oplus X$ as $\mathbb{C}V_{\mathbb{R}}$-modules. Namely, there is a matrix

$$P = \begin{pmatrix} I_{U_{\mathbb{R}}} & A \\ 0 & B \end{pmatrix}$$

such that $PY(v,z)P^{-1}$ is a diagonal matrix

$$\begin{pmatrix} Y^1(v,z) & 0 \\ 0 & Y^4(v,z) \end{pmatrix}$$

with $Y^4(v,z) \in \mathrm{End}(\mathbb{C}U_{\mathbb{R}}/\mathbb{C}W_{\mathbb{R}})[[z,z^{-1}]]$, where $I_{W_{\mathbb{R}}}$ is an identity of $\mathrm{End}(\mathbb{C}W_{\mathbb{R}})$, $A \in \mathrm{Hom}(\mathbb{C}U_{\mathbb{R}}/\mathbb{C}W_{\mathbb{R}}, \mathbb{C}W_{\mathbb{R}})$, and $B \in \mathrm{End}(\mathbb{C}U_{\mathbb{R}}/\mathbb{C}W_{\mathbb{R}})$. Set $A = A^1 + \sqrt{-1}A^2$ with real matrices A^i $(i = 1,2)$. By a direct calculation, $Y^1(v,z)AB^{-1} - Y^2(v,z)B^{-1} - AY^3(v,z)B^{-1} = 0$ and so we have $Y^1(v,z)A - Y^2(v,z) - AY^3(v,z) = 0$ and $Y^1(v,z)A^1 - Y^2(v,z) - A^1Y^3(v,z) = 0$. So if

$$Q = \begin{pmatrix} I_{W_{\mathbb{R}}} & A^1 \\ 0 & I_{U/W} \end{pmatrix}$$

with an identity map $I_{U_{\mathbb{R}}/W_{\mathbb{R}}}$ on $U_{\mathbb{R}}/W_{\mathbb{R}}$, then $QY(v,z)Q^{-1}$ is a diagonal matrix

$$\begin{pmatrix} Y^1(v,z) & 0 \\ 0 & Y^3(v,z) \end{pmatrix},$$

which contradicts to the choice of $U_{\mathbb{R}}$. ∎

About the fusion rules, it is easy to get the following:

LEMMA 2.3. *Let $W^1_{\mathbb{R}}$, $W^2_{\mathbb{R}}$, $W^3_{\mathbb{R}}$ be $V_{\mathbb{R}}$-modules. Then*

$$\dim I_{V_{\mathbb{R}}}\begin{pmatrix} & W^3_{\mathbb{R}} & \\ W^1_{\mathbb{R}} & & W^2_{\mathbb{R}} \end{pmatrix} \leq \dim I_{\mathbb{C}V_{\mathbb{R}}}\begin{pmatrix} & \mathbb{C}W^3_{\mathbb{R}} & \\ \mathbb{C}W^1_{\mathbb{R}} & & \mathbb{C}W^2_{\mathbb{R}} \end{pmatrix}.$$

Proof. It is easily to see that if $\{I^1, \ldots, I^k\}$ is a basis of

$$I_{V_{\mathbb{R}}}\begin{pmatrix} & W^3_{\mathbb{R}} & \\ W^1_{\mathbb{R}} & & W^2_{\mathbb{R}} \end{pmatrix}$$

over $\mathbb{R}$ then it is a linearly independent subset of

$$I_{\mathbb{C}V_{\mathbb{R}}}\begin{pmatrix} & \mathbb{C}W^3_{\mathbb{R}} & \\ \mathbb{C}W^1_{\mathbb{R}} & & \mathbb{C}W^2_{\mathbb{R}} \end{pmatrix}$$

over $\mathbb{C}$ by extending I^i to

$$\tilde{I}^i \in I_{\mathbb{C}V_{\mathbb{R}}}\begin{pmatrix} & \mathbb{C}W^3_{\mathbb{R}} & \\ \mathbb{C}W^1_{\mathbb{R}} & & \mathbb{C}W^2_{\mathbb{R}} \end{pmatrix}$$

by $\tilde{I}^i(u + \sqrt{-1}v, z) = I^i(u,z) + \sqrt{-1}I^i(v,z)$. ∎

3. $W(0)_{\mathbb{R}}$-MODULES AND FUSION RULES

We will next realize $L(c,h)_{\mathbb{R}}$ in a lattice VOA over $\mathbb{R}$ and its modules.

In order to get the upper bound of the coefficients in the fusion rules of $W(0)$-modules, the following will play an important role, which is a modified version of [DLe, Proposition 11.9].

THEOREM 3.1 [DLe]. *Let W^1, W^2, W^3 be V-modules and let $v_1 \in W^1$, $v_2 \in W^2$. Assume that W^1 and W^2 have no proper submodules containing v^1 and v^2, respectively. Let*

$$I(\cdot, z) \in I\begin{pmatrix} & W^3 & \\ W^1 & & W^2 \end{pmatrix}.$$

If $I(v^1, z)v^2 = 0$, then $I(\cdot, z) = 0$.

In the case where $W^1 = W(h,+) \oplus W(h,-)$ for $h = \frac{1}{15}, \frac{2}{3}$ and $0 \neq v \in W(h,+)$, W^1 has no proper $W(0)$-submodule containing $U^1 = \{(v, \phi(v)) \in W(h,+) \oplus W(h,-)\}$ since $W(h,+) \not\cong W(h,-)$ as $W(0)$-modules, where ϕ: $W(h,+) \to W(h,-)$ is an $L(\frac{4}{5},0)$-isomorphism. Using Theorem 3.1, we have an upper bound of the coefficients of the fusion rules as follows:

LEMMA 3.1. *The maps*

$$\phi^1\colon I_{W(0)}\begin{pmatrix} W^3 \\ W(i), W(j) \end{pmatrix} \to I_{L(4/5,0)}\begin{pmatrix} W^3 \\ L\left(\frac{4}{5}, i\right), L\left(\frac{4}{5}, j\right) \end{pmatrix},$$

$$\phi^2\colon I_{W(0)}\begin{pmatrix} W^3 \\ W(h,+) \oplus W(h,-), W(k,+) \oplus W(k,-) \end{pmatrix}$$

$$\to L_{L(4/5,0)}\begin{pmatrix} W^3 \\ L\left(\frac{4}{5}, h\right), L\left(\frac{4}{5}, k\right) \end{pmatrix}$$

and

$$\phi^3\colon I_{W(0)}\begin{pmatrix} W^3 \\ W(h,+) \oplus W(h,-), W(i) \end{pmatrix} \to I_{L(4/5,0)}\begin{pmatrix} W^3 \\ L\left(\frac{4}{5}, h\right), L\left(\frac{4}{5}, j\right) \end{pmatrix}$$

induced by the restrictions are all injective for $i, j = 0, \frac{2}{5}$ and $h, k = \frac{2}{3}, \frac{1}{15}$.

Throughout this paper, $N_{W^1, W^2}^{W^3}$ denotes

$$\dim I\begin{pmatrix} & W^3 & \\ W^1, & & W^2 \end{pmatrix}.$$

It is known that

$$N_{W^1, W^2}^{W^3} = N_{W^2, W^1}^{W^3} = N_{W^2, (W^3)'}^{(W^1)'},$$

where W' denotes the contragredient (dual) module of W (see [FrHLe, Proposition 5.5.2]). We note that $N_{W(0), W^1}^{W^1} = 1$ and $N_{W(0), W^1}^{W^2} = 0$ for $W^1 \not\cong W^2$. Let k' denote 3, $\frac{7}{5}$ for $k = 0, \frac{2}{5}$, respectively. Then $W(k) \cong L(\frac{4}{5}, k) \oplus L(\frac{4}{5}, k')$ as $L(\frac{4}{5}, 0)$-modules. By the above lemma and the fusion rules of the irreducible $L(\frac{4}{5}, 0)$-modules (see Table A), we have the following lemma.

LEMMA 3.2.

$$N_{W(i), W(j)}^{W(k)} \leq N_{L(4/5, i), L(4/5, j)}^{(4/5, k) \oplus L(4/5, k')} \leq 1,$$

$$N_{W(i), W(j)}^{W(k, \pm)} \leq N_{L(4/5, i), L(4/5, j)}^{L(4/5, k)} = 0,$$

$$N_{W(i, +) \oplus W(i, -), W(j, +) \oplus W(j, -)}^{W(k)} \leq N_{L(4/5, i), L(4/5, j)}^{L(4/5, k) \oplus L(4/5, k')} \leq 2,$$

$$N_{W(i, +) \oplus W(i, -), W(j, +) \oplus W(j, -)}^{W(k, \pm)} \leq N_{L(4/5, i), L(4/5, j)}^{L(4/5, k)} \leq 1.$$

Let us explain how we will determine the fusion rules of $W(0)$-modules. Let W^1 and W^2 be $W(0)$-modules. To simplify the notation, $W^1 \times W^2$ denotes a direct sum of irreducible $W(0)$-modules W^i with multiplicities λ_i if we have a fusion rule $W^1 \times W^2 = \sum \lambda_i W^i$. Let T be a sub VOA of $W(0)$ isomorphic to $L(\frac{4}{5}, 0)$. For a $W(0)$-module W, we denote it by W_T when we consider it as a T-module. Viewing W^1 and W^2 as T-modules, we have a fusion product $(W^1)_T \times (W^2)_T$. The above two lemmas tell that there is an injective T-homomorphism π of $(W^1 \times W^2)_T$ into $(W^1)_T \times (W^2)_T$. We note that all fusion rules $N_{L(4/5, j), L(4/5, h)}^{L(4/5, i)}$ are less than or equal to 1. We will next show $N_{W^2, W^3}^{W^1} \neq 0$ for the desired irreducible $W(0)$-modules W^1, W^2, W^3 so that the above injection π of $(W^1 \times W^2)_T$ into $(W^1)_T \times (W^2)_T$ is an isomorphism. We are therefore able to determine the fusion rules explicitly.

Let L be a lattice type $\sqrt{2}A_2$. Namely, set $L = \mathbb{Z}x + \mathbb{Z}y$ and $\langle x, x \rangle = \langle y, y \rangle = 4$ and $\langle x, y \rangle = -2$. In [FrLeMe], they construct a VOA $(V_N, Y, \mathbf{e}^0, \omega)$ from an even positive definite lattice N. It is easy to follow their construction over $\mathbb{R}$ and we have a VOA with Fock space $(V_N)_{\mathbb{R}} = \oplus_{a \in N} M(1)\mathbf{e}^a$ and a vertex operator $Y(v, z) = \sum_{i \in \mathbb{Z}} v_i z^{-i-1} \in \mathrm{End}((V_N)_{\mathbb{R}})[[z, z^{-1}]]$ of $v \in (V_N)_{\mathbb{R}}$, where $M(1) = \langle a(-n) : n = 1, 2, \dots \mid a \in \mathbb{R}N \rangle$. We also use the same notation for a vertex operator for modules. In [KMiY], they studied the structure of the VOA $M^0 = V_L$ and its modules $V_{\pm(x-y)/3+L}$. We will follow their way over $\mathbb{R}$. It is easy to see that $M_{\mathbb{R}}^1 = (V_{(x-y)/3+L})_{\mathbb{R}}$ and $M_{\mathbb{R}}^2 = (V_{(-x+y)/3+L})_{\mathbb{R}}$ are $(V_L)_{\mathbb{R}}$-modules. Set

$$M_{\mathbb{R}} = M_{\mathbb{R}}^0 \oplus M_{\mathbb{R}}^1 \oplus M_{\mathbb{R}}^2,$$

where $M^0_{\mathbb{R}} = (V_L)_{\mathbb{R}}$. We note that $M_{\mathbb{R}}$ is closed under the operators u_n of $u \in M_{\mathbb{R}}$. It is proved by [DLiMaN] that the Virasoro element w of $(V_L)_{\mathbb{R}}$ is an orthogonal sum of three conformal vectors w^1, w^2, and w^3 with central charges $\frac{1}{2}$, $\frac{7}{10}$, and $\frac{4}{5}$, respectively. Namely, $(V_L)_{\mathbb{R}}$ contains a sub VOA $T_{\mathbb{R}}$ isomorphic to $L(\frac{1}{2},0)_{\mathbb{R}} \otimes L(\frac{7}{10},0)_{\mathbb{R}} \otimes L(\frac{4}{5},0)_{\mathbb{R}}$. We note that each ω^i acts on $(V_S)_{\mathbb{R}}$ diagonally. Viewing $V_{\mathbb{R}}$ as a $T_{\mathbb{R}}$-module, it is a direct sum of irreducible modules of $T_{\mathbb{R}}$ and it follows from Lemma 2.2, Theorem 2.2, and [DMaZh] that each irreducible $T_{\mathbb{R}}$-module is isomorphic to $L(\frac{1}{2},h_1)_{\mathbb{R}} \otimes L(\frac{7}{10},h_2)_{\mathbb{R}} \otimes L(\frac{4}{5},h_3)_{\mathbb{R}}$ for some $h_1,h_2,h_3 \in \mathbb{R}$.

It comes from the construction of $(V_L)_{\mathbb{R}}$ that $((V_L)_{\mathbb{R}})_1 = \mathbb{R}x(-1)\mathbf{e}^0 + \mathbb{R}y(-1)\mathbf{e}^0$. The sum of all T-submodules of $M^0_{\mathbb{R}} = (V_L)_{\mathbb{R}}$ isomorphic to $L(\frac{1}{2},0)_{\mathbb{R}} \otimes L(\frac{7}{10},k_1)_{\mathbb{R}} \otimes L(\frac{4}{5},k_2)_{\mathbb{R}}$ for some k_1,k_2 is isomorphic to a direct sum of

$$T^1_{\mathbb{R}} = L\left(\frac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\frac{7}{10},0\right)_{\mathbb{R}} \otimes \left(L\left(\frac{4}{5},0\right)_{\mathbb{R}} \oplus L\left(\frac{4}{5},3\right)_{\mathbb{R}}\right)$$

and

$$T^2_{\mathbb{R}} = L\left(\frac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\frac{7}{10},\frac{3}{5}\right)_{\mathbb{R}} \otimes \left(L\left(\frac{4}{5},\frac{2}{5}\right)_{\mathbb{R}} \oplus L\left(\frac{4}{5},\frac{7}{5}\right)_{\mathbb{R}}\right).$$

Hence, $(V_L)_{\mathbb{R}}$ contains a sub VOA $\mathbf{1} \otimes \mathbf{1} \otimes L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}} \cong L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}$ and its module $L(\frac{4}{5},\frac{2}{5})_{\mathbb{R}} \oplus L(\frac{4}{5},\frac{7}{5})_{\mathbb{R}}$, which are denoted by $W(0)_{\mathbb{R}}$ and $W(\frac{2}{5})_{\mathbb{R}}$, respectively. Also, the sum of subspaces of $M^1_{\mathbb{R}}$ isomorphic to $L(\frac{1}{2},0)_{\mathbb{R}} \otimes L(\frac{7}{10},k_1)_{\mathbb{R}} \otimes L(\frac{4}{5},k_2)_{\mathbb{R}}$ is a direct sum of

$$T^{3+}_{\mathbb{R}} = L\left(\frac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\frac{7}{10},0\right)_{\mathbb{R}} \otimes L\left(\frac{4}{5},\frac{2}{3}\right)_{\mathbb{R}}$$

and

$$T^{4+}_{\mathbb{R}} = L\left(\frac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\frac{7}{10},\frac{3}{5}\right)_{\mathbb{R}} \otimes L\left(\frac{4}{5},\frac{1}{15}\right)_{\mathbb{R}}.$$

We hence have $W(0)_{\mathbb{R}}$-modules $L(\frac{4}{5},\frac{2}{3})_{\mathbb{R}}$ and $L(\frac{4}{5},\frac{1}{15})_{\mathbb{R}}$ denoted by $W(\frac{2}{3},+)_{\mathbb{R}}$ and $W(\frac{1}{15},+)_{\mathbb{R}}$, respectively.

Similarly, the sum of subspaces of $M^2_{\mathbb{R}}$ isomorphic to $L(\frac{1}{2},0)_{\mathbb{R}} \otimes L(\frac{7}{10},k_1)_{\mathbb{R}} \otimes L(\frac{4}{5},k_2)_{\mathbb{R}}$ is a direct sum of

$$T^{3-}_{\mathbb{R}} = L\left(\frac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\frac{7}{10},0\right)_{\mathbb{R}} \otimes L\left(\frac{4}{5},\frac{2}{3}\right)_{\mathbb{R}}$$

and

$$T^{4-}_{\mathbb{R}} = L\left(\frac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\frac{7}{10},\frac{3}{5}\right)_{\mathbb{R}} \otimes L\left(\frac{4}{5},\frac{1}{15}\right)_{\mathbb{R}}.$$

We hence obtain $W(0)_{\mathbb{R}}$-modules $L(\frac{4}{5}, \frac{2}{3})_{\mathbb{R}}$ and $L(\frac{4}{5}, \frac{1}{15})_{\mathbb{R}}$ denoted by $W(\frac{2}{3}, -)_{\mathbb{R}}$ and $W(\frac{1}{15}, -)_{\mathbb{R}}$, respectively.

It is easy to see that $T_{\mathbb{R}}^{n\,\pm}$ are the contragredient (dual) modules of $T_{\mathbb{R}}^{n\,\mp}$ by the natural invariant bilinear form on $V_{\mathbb{Z}((x+2y)/3)+L}$ given by $\langle e^a, e^b\rangle = \delta_{a,-b}$. By the construction, we have $\mathbb{C}W(h, \pm)_{\mathbb{R}} \cong W(h, \pm)$ and $\mathbb{C}W(h) \cong W(h)$ as $W(0)$-modules. By Theorems 1.1, 2.1, and 2.2, we have the following theorem:

THEOREM 3.2. *There is a VOA $W(0)_{\mathbb{R}}$ over $\mathbb{R}$ satisfying*

(1) $\mathbb{C}W(0)_{\mathbb{R}} \cong W(0)$,

(2) *$W(0)_{\mathbb{R}}$ is a rational VOA, and*

(3) *it has exactly the following six irreducible modules*:

$$W(0)_{\mathbb{R}}, W\left(\frac{2}{5}\right)_{\mathbb{R}}, W\left(\frac{2}{3}, +\right)_{\mathbb{R}}, W\left(\frac{1}{15}, +\right)_{\mathbb{R}}, W\left(\frac{2}{3}, -\right)_{\mathbb{R}}, W\left(\frac{1}{15}, -\right)_{\mathbb{R}},$$

where $\mathbb{C}W(h)_{\mathbb{R}} = W(h)$ and $\mathbb{C}W(h, \pm)_{\mathbb{R}} = W(h, \pm)$.

For $v \in \mathbb{R}L$, the definition of $Y(v, z)$ given in [FrLeMe] is general and we may view $Y(v, z) \in \mathrm{End}((V_{\mathbb{R}L})_{\mathbb{R}})\{z\}_{\mathbb{R}}$, where $R\{z\}_{\mathbb{R}}$ denotes a set $\{\sum_{i\in\mathbb{R}} a_i z^i \mid a_i \in R\}$. Since $\langle L, \frac{x+2y}{3} + L\rangle \subseteq 2\mathbb{Z}$, the vertex operators $Y(v, z)$ and $Y(u, z)$ satisfy the local commutativity for $v \in M^h$ and $u \in V_L$ and so we have an intertwining operator

$$Y(*, z)_{|M^g} \in I\begin{pmatrix} & M^{h+g} & \\ M^h, & & M^g \end{pmatrix}$$

by restricting on M^g.

By restriction, it is possible to view these operators as intertwining operators among $L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes \{L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)\}$-submodules and we also use them as intertwining operators among $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$-modules. At first, we obtain the following theorem by exactly the same arguments as in [Mi].

THEOREM 3.3. *Let $W_{\mathbb{R}}$ be a sub VOA of $V_{\mathbb{R}}$ with the Virasoro element $e \in (V_{\mathbb{R}})_2$. Assume that $W_{\mathbb{R}}$ is rational and e_1 acts on $V_{\mathbb{R}}$ diagonally. Let $M_{\mathbb{R}}^1$ and $M_{\mathbb{R}}^2$ be irreducible $W_{\mathbb{R}}$-submodules of $V_{\mathbb{R}}$. Set*

$$M\left(M_{\mathbb{R}}^1, M_{\mathbb{R}}^2\right) = \sum_{v\in M_{\mathbb{R}}^1,\, u\in M_{\mathbb{R}}^2,\, m\in\mathbb{Z}} \mathbb{R}v(m)u.$$

Then $M(M_{\mathbb{R}}^1, M_{\mathbb{R}}^2)$ is a $W_{\mathbb{R}}$-module and

$$I_W\begin{pmatrix} & M_{\mathbb{R}}^3 & \\ M_{\mathbb{R}}^1 & & M_{\mathbb{R}}^2 \end{pmatrix} \neq 0$$

for any irreducible $W_{\mathbb{R}}$-submodule $M^3_{\mathbb{R}}$ of $M(M^1_{\mathbb{R}}, M^2_{\mathbb{R}})$. Let $\{W^i_{\mathbb{R}} : i \in I\}$ be the set of distinct irreducible $W_{\mathbb{R}}$-modules and let $V_{\mathbb{R}} = \oplus V^i_{\mathbb{R}}$ be the decomposition into the direct sum of homogeneous $W_{\mathbb{R}}$-modules $V^i_{\mathbb{R}}$, where $V^i_{\mathbb{R}}$ is the sum of all irreducible $W_{\mathbb{R}}$-submodules of $V_{\mathbb{R}}$ isomorphic to $W^i_{\mathbb{R}}$. For $u \in V_{\mathbb{R}}$, we use the notation $u = \sum u^i$ with $u^i \in V^i_{\mathbb{R}}$. If there are $v \in V^i_{\mathbb{R}}$, $u \in V^j_{\mathbb{R}}$, $n \in \mathbb{Z}$ such that $(v_n u)^k \neq 0$, then

$$I\begin{pmatrix} & V^k_{\mathbb{R}} & \\ V^i_{\mathbb{R}}, & & V^j_{\mathbb{R}} \end{pmatrix} \neq 0.$$

We should note that $T^1_{\mathbb{R}} \oplus T^2_{\mathbb{R}} \oplus T^{3+}_{\mathbb{R}} \oplus T^{3-}_{\mathbb{R}} \oplus T^{4+}_{\mathbb{R}} \oplus T^{4-}_{\mathbb{R}}$ is closed under the products by Theorem 2.2 and the fusion rule $L(\frac{1}{2},0)_{\mathbb{R}} \times L(\frac{1}{2},0)_{\mathbb{R}} = L(\frac{1}{2},0)_{\mathbb{R}}$.

Since $W(h,-)_{\mathbb{R}}$ and $W(i)_{\mathbb{R}}$ are the contragredient (dual) modules of $W(h,+)_{\mathbb{R}}$ and $W(i)_{\mathbb{R}}$, respectively, we have

$$N^{W(0)_{\mathbb{R}}}_{W(h,\pm)_{\mathbb{R}}, W(h,\pm)_{\mathbb{R}}} \neq 0 \qquad \text{and} \qquad N^{W(0)_{\mathbb{R}}}_{W(i)_{\mathbb{R}}, W(i)_{\mathbb{R}}} \neq 0 \tag{3.1}$$

for $h = 0, \frac{2}{5}$ and $i = \frac{2}{3}, \frac{1}{15}$.

It is easy to check that

$$\begin{aligned}
\left(T^2_{\mathbb{R}}\right)_1 &= \mathbb{R}(x+2y)(-1)\mathbf{e}^0\\
\left(T^1_{\mathbb{R}}\right)_2 &= \mathbb{R}\left(3x(-1)^2\mathbf{e}^0 + (x+2y)(-1)^2\mathbf{e}^0\right)\\
\left(T^{3+}_{\mathbb{R}}\right)_{2/3} &= \mathbb{R}\left(\mathbf{e}^{(x+2y)/3} + \mathbf{e}^{(x-y)/3} + \mathbf{e}^{(-2x-y)/3}\right)\\
\left(T^{4+}_{\mathbb{R}}\right)_{2/3} &= \mathbb{R}\left(2\mathbf{e}^{(x+2y)/3} - \mathbf{e}^{(x-y)/3} - \mathbf{e}^{(-2x-y)/3}\right).
\end{aligned}$$

Since

$$\left((x+2y)(-1)\mathbf{e}^0\right)_{-1}(x+2y)(-1)\mathbf{e}^0 = (x+2y)(-1)^2\mathbf{e}^0 \notin (T^1)_2,$$

$$N^{W(2/5)_{\mathbb{R}}}_{W(2/5)_{\mathbb{R}}, W(2/5)_{\mathbb{R}}} \neq 0. \tag{3.2}$$

Also, since

$$\begin{aligned}
&(x+2y)(-1)\left\{\lambda\mathbf{e}^{(x+2y)/3} + \mu\left(\mathbf{e}^{(x-y)/3} + \mathbf{e}^{(-2x-y)/3}\right)\right\}\\
&\quad = 4\lambda\mathbf{e}^{(x+2y)/3} - 2\mu\left(\mathbf{e}^{(x+2y)/3} + \mathbf{e}^{(-2x-y)/3}\right),
\end{aligned}$$

we obtain

$$N^{W(1/15,+)_{\mathbb{R}}}_{W(2/5)_{\mathbb{R}}, W(2/3,+)_{\mathbb{R}}} = N^{W(2/5)_{\mathbb{R}}}_{W(2/3,+)_{\mathbb{R}}, W(1/15,-)_{\mathbb{R}}} = N^{W(2/3,-)_{\mathbb{R}}}_{W(2/5)_{\mathbb{R}}, W(1/15,-)_{\mathbb{R}}} \neq 0 \tag{3.3}$$

and

$$N^{W(1/15,+)_{\mathbb{R}}}_{W(2/5,)_{\mathbb{R}}, W(1/15,+)_{\mathbb{R}}} = N^{W(2/5)_{\mathbb{R}}}_{W(1/15,+)_{\mathbb{R}}, W(1/15,-)_{\mathbb{R}}} \neq 0. \tag{3.4}$$

Similarly,

$$N^{W(1/15,\,-)_{\mathbb{R}}}_{W(2/5)_{\mathbb{R}},\,W(2/3,\,-)_{\mathbb{R}}} = N^{W(2/5)_{\mathbb{R}}}_{W(2/3,\,-)_{\mathbb{R}},\,W(1/15,\,+)_{\mathbb{R}}} = N^{W(2/3,\,+)_{\mathbb{R}}}_{W(2/5)_{\mathbb{R}},\,W(1/15,\,+)_{\mathbb{R}}} \neq 0 \quad (3.5)$$

and

$$N^{W(1/15,\,-)_{\mathbb{R}}}_{W(2/5)_{\mathbb{R}},\,W(1/15,\,-)_{\mathbb{R}}} = N^{W(2/5)_{\mathbb{R}}}_{W(1/15,\,-)_{\mathbb{R}},\,W(1/15,\,+)_{\mathbb{R}}} \neq 0. \quad (3.6)$$

It follows from the direct calculcations that

$$\mathbf{e}^{(x+2y)/3} + \mathbf{e}^{(x-y)/3} + \mathbf{e}^{(-2x-y)/3} \quad \text{and}$$
$$\times 2\mathbf{e}^{(x+2y)/3} - (\mathbf{e}^{(x-y)/3} + \mathbf{e}^{(-2x-y)/3})$$

are highest weight vectors of

$$L\left(\tfrac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\tfrac{7}{10},0\right)_{\mathbb{R}} \otimes L\left(\tfrac{4}{5},\tfrac{2}{3}\right)_{\mathbb{R}} \subseteq M^1_{\mathbb{R}} \quad \text{and}$$
$$L\left(\tfrac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\tfrac{7}{10},\tfrac{3}{5}\right)_{\mathbb{R}} \otimes L\left(\tfrac{4}{5},\tfrac{1}{15}\right)_{\mathbb{R}} \subseteq M^1_{\mathbb{R}},$$

respectively. Similarly,

$$\mathbf{e}^{(-x-2y)/3} + \mathbf{e}^{(-x+y)/3} + \mathbf{e}^{(2x+y)/3} \quad \text{and}$$
$$2\mathbf{e}^{(-x-2y)/3} - (\mathbf{e}^{(-x+y)/3} + \mathbf{e}^{(2x+y)/3})$$

are highest weight vectors of

$$L\left(\tfrac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\tfrac{7}{10},0\right)_{\mathbb{R}} \otimes L\left(\tfrac{4}{5},\tfrac{2}{3}\right)_{\mathbb{R}} \subseteq M^2_{\mathbb{R}} \quad \text{and}$$
$$L\left(\tfrac{1}{2},0\right)_{\mathbb{R}} \otimes L\left(\tfrac{7}{10},\tfrac{3}{5}\right)_{\mathbb{R}} \otimes L\left(\tfrac{4}{5},\tfrac{1}{15}\right)_{\mathbb{R}} \subseteq M^2_{\mathbb{R}},$$

respectively. Also, for

$$u = \alpha\mathbf{e}^{(x+2y)/3} + \beta(\mathbf{e}^{(x-y)/3} + \mathbf{e}^{(-2x-y)/3}) \quad \text{and}$$
$$v = \lambda\mathbf{e}^{(x+2y)/3} + \mu(\mathbf{e}^{(x-y)/3} + \mathbf{e}^{(-2x-y)/3}),$$

we obtain $u_{-1/3}v = 2\beta\mu\mathbf{e}^{(x+2y)/3} + \beta\lambda + \alpha\mu(\mathbf{e}^{(x-y)/3} + \mathbf{e}^{(-2x-y)/3})$, where $u_{-1/3}$ is the grade preserving operator of u. Hence, we have

$$N^{W(2/3,\,\mp)_{\mathbb{R}}}_{W(2/3,\,\pm)_{\mathbb{R}},\,W(2/3,\,\pm)_{\mathbb{R}}} \neq 0, \quad (3.7)$$

$$N^{W(2/3,\,\mp)_{\mathbb{R}}}_{W(1/15,\,\pm)_{\mathbb{R}},\,W(1/15,\,\pm)_{\mathbb{R}}} = N^{W(1/15,\,\mp)_{\mathbb{R}}}_{W(1/15,\,\pm)_{\mathbb{R}},\,W(2/3,\,\pm)_{\mathbb{R}}} \neq 0, \quad (3.8)$$

$$N^{W(1/15,\,\mp)_{\mathbb{R}}}_{W(1/15,\,\pm)_{\mathbb{R}},\,W(1/15,\,\pm)_{\mathbb{R}}} \neq 0. \quad (3.9)$$

4. FUSION RULE

We first list the fusion rules among $L(\frac{4}{5},0)$-modules $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},\frac{2}{5})$, $L(\frac{4}{5},\frac{7}{5})$, $L(\frac{4}{5},\frac{2}{3})$, and $L(\frac{4}{5},\frac{1}{15})$, which are the only irreducible modules we need in this section. For the fusion rules for the remaining cases, see the Appendix.

Table A

0	$\frac{2}{5}$	$\frac{7}{5}$	$\frac{1}{15}$	3	$\frac{2}{3}$
$\frac{2}{5}$	$0:\frac{7}{5}$	$\frac{2}{5}:3$	$\frac{1}{15}:\frac{2}{3}$	$\frac{7}{5}$	$\frac{1}{15}$
$\frac{7}{5}$	$\frac{2}{5}:3$	$0:\frac{7}{5}$	$\frac{2}{3}:\frac{1}{15}$	$\frac{2}{5}$	$\frac{1}{15}$
$\frac{1}{15}$	$\frac{1}{15}:\frac{2}{3}$	$\frac{2}{3}:\frac{1}{15}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}:3:\frac{2}{5}$	$\frac{1}{15}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$
3	$\frac{7}{5}$	$\frac{2}{5}$	$\frac{1}{15}$	0	$\frac{2}{3}$
$\frac{2}{3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$	$\frac{2}{3}$	$0:\frac{2}{3}:3$

In the table, the number h denotes $L(\frac{4}{5},h)$ and $h:\cdots:k$ denotes $L(\frac{4}{5},h) + \cdots + L(\frac{4}{5},k)$. We note that if $L(\frac{4}{5},h)\times L(\frac{4}{5},k) = \sum \lambda_i L(\frac{4}{5},h_i)$ and $L(\frac{4}{5},h)_{\mathbb{R}} \times L(\frac{4}{5},k)_{\mathbb{R}} = \sum \mu_i L(\frac{4}{5},h_i)_{\mathbb{R}}$, then $\mu_i \le \lambda_i$ by Lemma 2.2.

By (3.5) and (3.6), $N_{W(h,\pm)_{\mathbb{R}},W(h,\pm)_{\mathbb{R}}}^{W(h,\mp)_{\mathbb{R}}} \neq 0$. Hence, it follows from the fusion rule $L(\frac{4}{5},\frac{2}{3})\times L(\frac{4}{5},\frac{2}{3}) = L(\frac{4}{5},0) + L(\frac{4}{5},3) + L(\frac{4}{5},\frac{2}{3})$ of $L(\frac{4}{5},0)$-modules and (3.1) and (3.7) that

$$\begin{aligned} W\left(\tfrac{2}{3},\pm\right)_{\mathbb{R}} \times W\left(\tfrac{2}{3},\pm\right)_{\mathbb{R}} &= W\left(\tfrac{2}{3},\mp\right)_{\mathbb{R}} \\ W\left(\tfrac{2}{3},\pm\right)_{\mathbb{R}} \times W\left(\tfrac{2}{3},\mp\right)_{\mathbb{R}} &= W(0)_{\mathbb{R}} \\ W\left(\tfrac{2}{3},\pm\right) \times W\left(\tfrac{2}{3},\pm\right) &= W\left(\tfrac{2}{3},\mp\right) \\ W\left(\tfrac{2}{3},\pm\right) \times W\left(\tfrac{2}{3},\mp\right) &= W(0). \end{aligned} \tag{4.1}$$

From now on, we will omit the fusion rules of $W(0)_{\mathbb{R}}$-modules because they will have the same form as the fusion rules of $W(0)$-modules.

Similarly, by the fusion rule $L(\frac{4}{5},\frac{1}{15})\times L(\frac{4}{5},\frac{1}{15}) = L(\frac{4}{5},0) + L(\frac{4}{5},3) + L(\frac{4}{5},\frac{2}{3}) + L(\frac{4}{5},\frac{1}{15}) + L(\frac{4}{5},\frac{2}{5}) + L(\frac{4}{5},\frac{7}{5})$ of $L(\frac{4}{5},0)$-modules and (3.1), (3.4), (3.8), and (3.9), we obtain

$$\begin{aligned} W\left(\tfrac{1}{15},\pm\right) \times W\left(\tfrac{1}{15},\pm\right) &= W\left(\tfrac{1}{15},\mp\right) + W\left(\tfrac{2}{3},\mp\right) \\ W\left(\tfrac{1}{15},\pm\right) \times W\left(\tfrac{1}{15},\mp\right) &= W(0) + W\left(\tfrac{2}{5}\right). \end{aligned} \tag{4.2}$$

The fusion rule $L(\frac{4}{5},\frac{2}{5}) \times L(\frac{4}{5},\frac{2}{3}) = L(\frac{4}{5},\frac{1}{15})$ and (3.3) and (3.5) imply that

$$W\left(\frac{2}{5}\right) \times W\left(\frac{2}{3}, \pm\right) = W\left(\frac{1}{15}, \pm\right). \tag{4.3}$$

By the fusion rule $L(\frac{4}{5},\frac{2}{5}) \times L(\frac{4}{5},\frac{1}{15}) = L(\frac{4}{5},\frac{1}{15}) + L(\frac{4}{5},\frac{2}{3})$ and (3.3)–(3.6), we have

$$W\left(\frac{2}{5}\right) \times W\left(\frac{1}{15}, \pm\right) = W\left(\frac{1}{15}, \pm\right) + W\left(\frac{2}{3}, \pm\right). \tag{4.4}$$

It follows from the fusion rule $L(\frac{4}{5},\frac{2}{5}) \times L(\frac{4}{5},\frac{2}{5}) = L(\frac{4}{5},0) + L(\frac{4}{5},\frac{7}{5})$ and (3.1) and (3.2) that

$$W\left(\frac{2}{5}\right) \times W\left(\frac{2}{5}\right) = W(0) + W\left(\frac{2}{5}\right). \tag{4.5}$$

By the fusion rule $L(\frac{4}{5},\frac{2}{3}) \times L(\frac{4}{5},\frac{1}{15}) = L(\frac{4}{5},\frac{2}{5}) + L(\frac{4}{5},\frac{7}{5}) + L(\frac{4}{5},\frac{1}{15})$ and (3.3), (3.5), and (3.8), we obtain

$$\begin{aligned} &W\left(\tfrac{2}{3}, \pm\right) \times W\left(\tfrac{1}{15}, \pm\right) = W\left(\tfrac{1}{15}, \mp\right) \qquad \text{and} \\ &W\left(\tfrac{2}{3}, \pm\right) \times W\left(\tfrac{1}{15}, \mp\right) = W\left(\tfrac{2}{5}\right). \end{aligned} \tag{4.6}$$

THEOREM 4.1. *By using the correspondence $W(h,*)_{\mathbb{R}} \Leftrightarrow W(h,*)$ between $W(0)_{\mathbb{R}}$-modules and $W(0)$-modules, the fusion rules for $W(0)_{\mathbb{R}}$-modules are equal to the fusion rules for $W(0)$-modules.*

We put the above fusion rules in the following table.

Table B

$W(0)$	$W(\frac{2}{5})$	$W(\frac{2}{3},+)$	$W(\frac{1}{15},+)$	$W(\frac{2}{3},-)$	$W(\frac{1}{15},-)$
$W(\frac{2}{5})$	$W(0):W(\frac{2}{5})$	$W(\frac{1}{15},+)$	$W(\frac{1}{15},+):W(\frac{2}{3},+)$	$W(\frac{1}{15},-)$	$W(\frac{1}{15},-):W(\frac{2}{3},-)$
$W(\frac{2}{3},+)$	$W(\frac{1}{15},+)$	$W(\frac{2}{3},-)$	$W(\frac{1}{15},-)$	$W(0)$	$W(\frac{2}{5})$
$W(\frac{1}{15},+)$	$W(\frac{1}{15},+):W(\frac{2}{3},+)$	$W(\frac{1}{15},-)$	$W(\frac{1}{15},-):W(\frac{2}{3},-)$	$W(\frac{2}{5})$	$W(0):W(\frac{2}{5})$
$W(\frac{2}{3},-)$	$W(\frac{1}{15},-)$	$W(0)$	$W(\frac{2}{5})$	$!W(\frac{2}{3},+)$	$W(\frac{1}{15},+)$
$W(\frac{1}{15},-)$	$W(\frac{1}{15},-):W(\frac{2}{3},-)$	$W(\frac{2}{5})$	$W(0):W(\frac{2}{5})$	$W(\frac{1}{15},+)$	$W(\frac{1}{15},+):W(\frac{2}{3},+)$

5. AUTOMORPHISMS OF VOAS

In the previous section we determined the fusion rules of $W(0)$-modules. As an application we will have the following theorem.

THEOREM 5.1. *If a VOA V contains a sub VOA X isomorphic to $W(0)$, then an endomorphism τ_X of V defined by*

$$\sigma_X : \begin{cases} 1 & \text{on } W(0) \quad \text{and} \quad W\left(\frac{2}{5}\right), \\ e^{2\pi i/3} & \text{on } W\left(\frac{2}{3}, +\right) \quad \text{and} \quad W\left(\frac{1}{15}, +\right), \\ e^{4\pi i/3} & \text{on } W\left(\frac{2}{3}, -\right) \quad \text{and} \quad W\left(\frac{1}{15}, -\right), \end{cases}$$

is an automorphism of V.

Proof. Replacing $W(i)$ and $W(h,+)$ and $W(k,-)$ in Table B by 1 and $e^{2\pi i/3}$ and $e^{4\pi i/3}$, respectively, we have

1	1	$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}$
1	$1:1$	$e^{2\pi i/3}$	$e^{2\pi i/3}:e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}:e^{4\pi i/3}$
$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}$	1	1
$e^{2\pi i/3}$	$e^{2\pi i/3}:e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}:e^{4\pi i/3}$	1	$1:1$
$e^{4\pi i/3}$	$e^{4\pi i/3}$	1	1	$e^{2\pi i/3}$	$e^{2\pi i/3}$
$e^{4\pi i/3}$	$e^{4\pi i/3}:e^{4\pi i/3}$	1	$1:1$	$e^{2\pi i/3}$	$e^{2\pi i/3}:e^{2\pi i/3}$

which is compatible with the products. Hence, σ_X is an automorphism of V by Theorem 3.3. ∎

If $\sigma_X = 1$ then all T-submodules of V are isomorphic to $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},\frac{2}{5})$, or $L(\frac{4}{5},\frac{7}{5})$ for $T \subseteq W$ with $T \cong L(\frac{4}{5},0)$. In this case we can define another automorphism μ_T of V as follows:

THEOREM 5.2. *Assume that V contains a sub VOA T isomorphic to $L(\frac{4}{5},0)$ and all T-submodules of V are isomorphic to $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},\frac{2}{5})$, or $L(\frac{4}{5},\frac{7}{5})$. Then an endomorphism μ_T of V defined by*

$$\mu_T : \begin{cases} 1 & \text{on } L\left(\frac{4}{5},0\right) \quad \text{and} \quad L\left(\frac{4}{5},\frac{7}{5}\right) \\ -1 & \text{on } L\left(\frac{4}{5},3\right) \quad \text{and} \quad L\left(\frac{4}{5},\frac{2}{5}\right) \end{cases}$$

is an automorphism of V.

6. AUTOMORPHISM OF VOA OVER THE REAL NUMBER FIELD

In this section, we will define an automorphism of a VOA $V_{\mathbb{R}}$ with a positive definite invariant bilinear form.

In Section 3, we constructed a VOA $W(0)_{\mathbb{R}}$ and its modules $W(\frac{2}{5})_{\mathbb{R}}$, $W(\frac{2}{3},\pm)_{\mathbb{R}}$, $W(\frac{1}{15},\pm)_{\mathbb{R}}$ by the same arguments as in [KMiY]. As they showed, there is a highest weight vector $q \in L(\frac{4}{5},3)_{\mathbb{R}}$ such that $q_5 q = \mathbf{1}$. In particular, $W(0)_{\mathbb{R}}$ does not have a positive definite invariant bilinear form. Let us show another VOA structure on $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ with a positive definite invariant bilinear form. Since $W(0)_{\mathbb{R}} = L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}$ is $\mathbb{Z}_2$-graded, i.e., $W(0)_{\mathbb{R}} = W^1 \oplus W^2$ with $W^0 = L(\frac{4}{5},0)_{\mathbb{R}}$ and $W^1 = L(\frac{4}{5},3)_{\mathbb{R}}$, we can write vertex operators of elements in $W(0)_{\mathbb{R}}$ by 2×2-matrices,

$$Y(u,z) = \begin{pmatrix} Y^{00}(u,z) & 0 \\ 0 & Y^{11}(u,z) \end{pmatrix} \qquad \text{for } u \in L\left(\tfrac{4}{5},0\right)_{\mathbb{R}}$$

$$Y(v,z) = \begin{pmatrix} 0 & Y^{10}(u,z) \\ Y^{01}(u,z) & 0 \end{pmatrix} \qquad \text{for } v \in L\left(\tfrac{4}{5},3\right)_{\mathbb{R}},$$

where $Y^{ij}(*,z) \in \operatorname{Hom}(W^i, W^j)[[z,z^{-1}]]$. Then

$$\tilde{Y}(u,z) = \begin{pmatrix} Y^{00}(u,z) & 0 \\ 0 & Y^{11}(u,z) \end{pmatrix} \qquad \text{for } u \in L\left(\tfrac{4}{5},0\right)_{\mathbb{R}}$$

$$\tilde{Y}(v,z) = \begin{pmatrix} 0 & -Y^{10}(u,z) \\ Y^{01}(u,z) & 0 \end{pmatrix} \qquad \text{for } v \in L\left(\tfrac{4}{5},3\right)_{\mathbb{R}}$$

offers another VOA structure on $L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}$. In this VOA, $q_5 q = -\mathbf{1}$ and it is not difficult to prove that this new VOA $(L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}, \tilde{Y})$ has a positive definite invariant bilinear form. We denote $(L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}, \tilde{Y})$ by $W(0)_{\mathbb{R}}^{+}$. There is another simple explanation. The new VOA $W(0)_{\mathbb{R}}^{+}$ is a sub VOA $L(\frac{4}{5},0)_{\mathbb{R}} + \sqrt{-1}\,L(\frac{4}{5},3)_{\mathbb{R}}$ of $\mathbb{C}W(0)_{\mathbb{R}} = \mathbb{C}(L(\frac{4}{5},0)_{\mathbb{R}} + L(\frac{4}{5},3)_{\mathbb{R}})$.

THEOREM 6.1. *$\mathbb{C}W(0)_{\mathbb{R}}^{+} \cong W(0)$ as VOAs over $\mathbb{C}$. $W(0)_{\mathbb{R}}^{+}$ is a rational VOA and it has exactly four irreducible modules*:

$$W(0)_{\mathbb{R}}^{+},\, W\left(\frac{2}{5}\right)_{\mathbb{R}}^{+},\, W\left(\frac{2}{3}\right)_{\mathbb{R}}^{+},\, W\left(\frac{1}{15}\right)_{\mathbb{R}}^{+}.$$

Proof. Set $W(0)_{\mathbb{R}} = L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}$. As we explained, a sub VOA $L(\frac{4}{5},0)_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},3)_{\mathbb{R}}$ of $\mathbb{C}W(0)_{\mathbb{R}}$ is isomorphic to $W(0)_{\mathbb{R}}^{+}$ as a VOA over $\mathbb{R}$. Let $M_{\mathbb{R}}$ be a $W(0)_{\mathbb{R}}^{+}$-module. Then $\mathbb{C}M_{\mathbb{R}}$ is an $W(0)$-module and so $\mathbb{C}M_{\mathbb{R}}$ is a direct sum of irreducible $W(0)$-modules. Let U be an irreducible $W(0)$-submodule of $\mathbb{C}M_{\mathbb{R}}$. If $U \cong \mathbb{C}W(h,\pm)_{\mathbb{R}}$ as $\mathbb{C}W(0)_{\mathbb{R}}$-modules, then $W(0)_{\mathbb{R}}^{+} = L(\frac{4}{5},0)_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},3)_{\mathbb{R}}$ implies that $\mathbb{C}W(h,\pm)_{\mathbb{R}}$ is an irreducible $W(0)_{\mathbb{R}}^{+}$-module. Since $\mathbb{C}W(h,\pm)_{\mathbb{R}} = L(\frac{4}{5},h)_{\mathbb{R}} \oplus L(\frac{4}{5},h)_{\mathbb{R}}$ as $L(\frac{4}{5},0)_{\mathbb{R}}$-modules and

$$\dim I\begin{pmatrix} & L\left(\tfrac{4}{5},h\right)_{\mathbb{R}} & \\ L\left(\tfrac{4}{5},3\right)_{\mathbb{R}}, & & L\left(\tfrac{4}{5},h\right)_{\mathbb{R}} \end{pmatrix} = 1,$$

we have $\mathbb{C}W(h,+)_{\mathbb{R}} \cong \mathbb{C}W(h,-)_{\mathbb{R}}$ as $W(0)^+_{\mathbb{R}}$-modules. If $U \cong \mathbb{C}W(h)_{\mathbb{R}} = \mathbb{C}L(\frac{4}{5},h)_{\mathbb{R}} \oplus \mathbb{C}L(\frac{4}{5},h')_{\mathbb{R}}$ as $\mathbb{C}W(0)_{\mathbb{R}}$-modules, then $\mathbb{C}W(h)_{\mathbb{R}} = L(\frac{4}{5},h)_{\mathbb{R}} \oplus L(\frac{4}{5},h')_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},h)_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},h')_{\mathbb{R}}$. Hence, $\mathbb{C}W(h)_{\mathbb{R}}$ is a direct sum of two irreducible $W(0)^+_{\mathbb{R}}$-modules $L(\frac{4}{5},h)_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},h')_{\mathbb{R}}$ and $\sqrt{-1}\,(L(\frac{4}{5},h)_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},h'))_{\mathbb{R}}$. Thus, $\mathbb{C}M_{\mathbb{R}}$ is a direct sum of irreducible $W(0)^+_{\mathbb{R}}$-modules. Since $\mathbb{C}M_{\mathbb{R}} \cong M_{\mathbb{R}} \oplus M_{\mathbb{R}}$ as $W(0)^+_{\mathbb{R}}$-modules, we obtain the desired results. ∎

LEMMA 6.1. *$\mathbb{C}W(h)^+_{\mathbb{R}} \cong W(h,+) \oplus W(h,-)$ as $W(0)$-modules for $h = \frac{2}{3}, \frac{1}{15}$.*

Proof. As we showed in the proof of Theorem 6.1, $\mathbb{C}W(h,\pm)_{\mathbb{R}} \cong W(h)^+_{\mathbb{R}}$ as $W(0)^+_{\mathbb{R}}$-modules. Since $\mathbb{C}W(h)^+_{\mathbb{R}} \cong L(\frac{4}{5},h)^{\oplus 4}_{\mathbb{R}}$ as $L(\frac{4}{5},0)_{\mathbb{R}}$-modules, $\mathbb{C}W(h)^+_{\mathbb{R}}$ is a direct sum of $W(h,+)$ and $W(h,-)$ as $W(0)$-modules. ∎

Since $W(h,+)_{\mathbb{R}} \cong W(h,-)_{\mathbb{R}}$ as $L(\frac{4}{5},0)_{\mathbb{R}}$-modules, there is an $L(\frac{4}{5},0)_{\mathbb{R}}$-isomorphism ϕ: $W(h,+)_{\mathbb{R}} \to W(h,-)_{\mathbb{R}}$. We fix it for a while.

LEMMA 6.2. *Let $U_{\mathbb{R}}$ be an irreducible $W(0)^+_{\mathbb{R}}$-submodule of $\mathbb{C}W(h,+)_{\mathbb{R}} \oplus \mathbb{C}W(h,-)_{\mathbb{R}}$. Then one of the followings holds*:

(1) $U_{\mathbb{R}} = \mathbb{C}W(h,+)_{\mathbb{R}}$.

(2) $U_{\mathbb{R}} = \mathbb{C}W(h,-)_{\mathbb{R}}$.

(3) *There is an $r \in \mathbb{C}$ such that*

$$U_{\mathbb{R}} = \left\{\left(kv, r\bar{k}\phi(v)\right) \in \mathbb{C}W(h,+)_{\mathbb{R}} \oplus \mathbb{C}W(h,-)_{\mathbb{R}} : k \in \mathbb{C}, v \in W(h,+)_{\mathbb{R}}\right\},$$

where $\bar{k}$ is the complex conjugate of k.

Proof. It is clear that $\mathrm{Hom}_{L(4/5,0)_{\mathbb{R}}}(L(\frac{4}{5},h)_{\mathbb{R}}, \mathbb{C}L(\frac{4}{5},h)_{\mathbb{R}}) = \mathbb{C}\phi$. We note $W(0)_{\mathbb{R}} = L(\frac{4}{5},0)_{\mathbb{R}} \oplus L(\frac{4}{5},3)_{\mathbb{R}}$ and $W(0)^+_{\mathbb{R}} = L(\frac{4}{5},0)_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},3)_{\mathbb{R}}$. If $U_{\mathbb{R}} \cap \mathbb{C}W(h,\pm)_{\mathbb{R}} \neq 0$, then $U_{\mathbb{R}} \cap \mathbb{C}W(h,\pm)_{\mathbb{R}}$ is invariant under the action of $L(\frac{4}{5},0)_{\mathbb{R}} \oplus \sqrt{-1}\,L(\frac{4}{5},3)_{\mathbb{R}}$ and so $U_{\mathbb{R}} = \mathbb{C}W(h,\pm)_{\mathbb{R}}$. So we may assume that $U_{\mathbb{R}} \cap \mathbb{C}W(h,+)_{\mathbb{R}} = U_{\mathbb{R}} \cap \mathbb{C}W(h,-)_{\mathbb{R}} = 0$. Then, $U_{\mathbb{R}} = \{(u, p(u)) : u \in \mathbb{C}W(h,+)_{\mathbb{R}},\ p(u) \in \mathbb{C}W(h,-)_{\mathbb{R}}\}$ for some map p: $\mathbb{C}W(h,+)_{\mathbb{R}} \to \mathbb{C}W(h,-)_{\mathbb{R}}$. Since p: $W(h,+)_{\mathbb{R}} \to \mathbb{C}W(h,-)_{\mathbb{R}}$ is an $L(\frac{4}{5},0)_{\mathbb{R}}$-homomorphism, there is a complex number $r \in \mathbb{C}$ such that $p(u) = r\phi(u)$ for $u \in W(h,+)_{\mathbb{R}}$. Let $q \in L(\frac{4}{5},3)_{\mathbb{R}}$ be a highest weight vector and let $Y^{\pm}(q,z) = \sum q_n^{\pm} z^{-n-1}$ be the module vertex operators of q on $W(h,\pm)_{\mathbb{R}}$. Then $Y^+(q,z) = -Y^-(q,z)$ by the constructions of $W(h,\pm)_{\mathbb{R}}$; see [KMiY]. It follows from $U_{\mathbb{R}} \ni (\sqrt{-1}\,q_n^+u, \sqrt{-1}\,q_n^- p(u)) = (\sqrt{-1}\,q_n^+u, \sqrt{-1}\,p(q_n^- u)) = (\sqrt{-1}\,q_n^+u, \sqrt{-1}\,p(q_n^+ u))$ that $p(\sqrt{-1}\,v) = -\sqrt{-1}\,p(v) = -\sqrt{-1}\,r\phi(v)$ for $v \in W(h,+)$. We hence have (3). ∎

Using $W(0)^+_{\mathbb{R}}$, we will define an automorphism of V.

THEOREM 6.2. *Assume that a VOA $V_{\mathbb{R}}$ contains a sub VOA $W_{\mathbb{R}} \cong W(0)^+_{\mathbb{R}}$; then the automorphism $\sigma_{\mathbb{C}W_{\mathbb{R}}}$ of $\mathbb{C}V_{\mathbb{R}}$ defined by $\mathbb{C}W_{\mathbb{R}}$ keeps $V_{\mathbb{R}}$ invariant. In particular, $\sigma_{W_{\mathbb{R}}} = \sigma_{\mathbb{C}W_{\mathbb{R}} \mid V_{\mathbb{R}}}$ is an automorphism of $V_{\mathbb{R}}$.*

Proof. Set $W_{\mathbb{R}} = L(\frac{4}{5}, 0)_{\mathbb{R}} \oplus L(\frac{4}{5}, 3)_{\mathbb{R}}$ and $\tilde{W}_{\mathbb{R}} = L(\frac{4}{5}, 0)_{\mathbb{R}} \oplus \sqrt{-1}\, L(\frac{4}{5}, 3)_{\mathbb{R}} \cong W(0)_{\mathbb{R}}$. Since $\mathbb{C}V_{\mathbb{R}}$ contains $\mathbb{C}W_{\mathbb{R}} \cong W(0)$, we have an automorphism $\sigma_{\mathbb{C}W_{\mathbb{R}}}$ by Theorem 5.1. Since $W(0)^+_{\mathbb{R}}$ is rational, $V_{\mathbb{R}}$ is a direct sum of irreducible $W(0)^+_{\mathbb{R}}$-modules. Let $U_{\mathbb{R}}$ be an irreducible $W_{\mathbb{R}}$-module of $V_{\mathbb{R}}$. Then $\mathbb{C}U_{\mathbb{R}}$ is a $\mathbb{C}\tilde{W}_{\mathbb{R}}$-module. If $U_{\mathbb{R}} \cong W(0)^+_{\mathbb{R}}$ or $W(\frac{2}{5})^+_{\mathbb{R}}$ as $W(0)^+_{\mathbb{R}}$-modules, then $\mathbb{C}U_{\mathbb{R}} \cong \mathbb{C}W(0)_{\mathbb{R}}$ or $\mathbb{C}W(\frac{2}{5})_{\mathbb{R}}$ as $\mathbb{C}W_{\mathbb{R}}$-modules. Hence, $\sigma_{\mathbb{C}W_{\mathbb{R}}}$ fixes all elements in $\mathbb{C}U_{\mathbb{R}}$ and so $\sigma_{\mathbb{C}W_{\mathbb{R}}}(U_{\mathbb{R}}) = U_{\mathbb{R}}$. If $U_{\mathbb{R}} \cong W(h)^+_{\mathbb{R}}$ as $W(0)^+_{\mathbb{R}}$-modules for $h = \frac{2}{3}$ or $\frac{1}{15}$ then $\mathbb{C}U_{\mathbb{R}} \cong \mathbb{C}W(h,+)_{\mathbb{R}} \oplus \mathbb{C}W(h,-)_{\mathbb{R}}$ as $\mathbb{C}W(0)_{\mathbb{R}}$-modules. Hence, $U_{\mathbb{R}}$ equals one of $\mathbb{C}W(h,+)_{\mathbb{R}}$, $\mathbb{C}W(h,-)_{\mathbb{R}}$, or $U_{\mathbb{R}} = \{(ku, \bar{k}r\phi(u)) : u \in W(h,+)_{\mathbb{R}}, k \in \mathbb{C}\}$ for some $r \in \mathbb{C}$. Since $\sigma_{\mathbb{C}W_{\mathbb{R}}}$ acts as

$$\begin{array}{ll} e^{2\pi i/3} & \text{on } \mathbb{C}L(h,+)_{\mathbb{R}} \\ e^{-2\pi i/3} & \text{on } \mathbb{C}L(h,-)_{\mathbb{R}} \end{array}$$

and

$$\sigma_{\mathbb{C}W_{\mathbb{R}}} \colon \big(ku, \bar{k}r\phi(u)\big) \to \big(e^{2\pi i/3}ku, e^{-2\pi i/3}\bar{k}r\phi(u)\big),$$

we obtain $\sigma_{\mathbb{C}W_{\mathbb{R}}}(U_{\mathbb{R}}) = U_{\mathbb{R}}$ in any case. Thus $\sigma_{\mathbb{C}W_{\mathbb{R}}}$ keeps $V_{\mathbb{R}}$ invariant. ∎

7. V AS A SUB VOA-MODULE

In this section, we assume that a VOA V is defined over a subfield K of $\mathbb{C}$. The notion of sub VOAs of V in this paper is not the same as in [FrZh], where they expected sub VOAs W to have the same Virasoro element with V. Our definition of sub VOAs is:

$W = (\oplus W_n, Y^W, e, \mathbf{1}_W)$ is a sub VOA of $V = (\oplus V_n, Y, \omega, \mathbf{1}_V)$ if

(1) $(W, Y^W, e, \mathbf{1}_W)$ is a VOA,

(2) $W \subseteq V$ and $W_n = W \cap V_n$,

(3) $\mathbf{1}_W = \mathbf{1}_V$, and

(4) $Y^W(v,z) = Y(v,z)\mid_W$ for $v \in W$.

In this section, we assume that K contains all eigenvalues of e_1 on $\mathbb{C}V$.

There are several definitions for VOA-modules, but we will include an infinite direct sum of irreducible modules as a VOA-module M. Different from the ordinary algebras, it is not generally true that V is a W-module. The problem is whether e_1 acts on V diagonally or not. The purpose of this section is to show that if W is rational, then V is a W-module. Set $f = w - e$. Since $e \in W_2 \subseteq V_2$, e_1 preserves the degree of V. By the assumption (2), $f_1 = \omega_1 - e_1$ acts on W as 0. Furthermore, for $v \in W$, we have $Y(e_0 v, z)\mid_W = \frac{d}{dz}Y(v,z)\mid_W = Y(\omega_0 v, z)\mid_W$ and so $Y(\omega_0 v - e_0 v, z)\mid_W = 0$. In particular, $\omega_0 v - e_0 v = (\omega_0 v - e_0 v)_{-1}\mathbf{1} = 0$ and so $\omega_0 v = e_0 v \in W$. Hence $f_0 = \omega_0 - e_0$ acts on W as 0.

LEMMA 7.1. *Both f_1 and f_0 commute v_m on V for any $v \in W$ and $m \in \mathbb{Z}$.*

Proof. This follows from $[f_0, v_m] = (f_0 v)_m = 0$ and $[f_1, v_m] = (f_0 v)_{m+1} + (f_1 v)_m = 0$. ∎

We will prove the following theorem:

THEOREM 7.1. *If W is rational, then V is a W-module.*

Proof. Clearly, vertex operators $\{Y(v,z) \mid v \in W\}$ satisfy the local commutativity, the associativity, and the e_0-derivative property:

$$Y(e_0 v, z) = Y\big((f_0 + e_0)v, z\big) = Y(\omega_0 v, z) = \frac{d}{dz}Y(v,z).$$

Hence, what we have to do is to prove that V is a direct sum of eigenspaces of e_1. Suppose this is false. We note that the eigenspace V_λ and the generalized eigenspace $T_\lambda = \{v \in V \mid \exists n \in \mathbb{Z}(f_1 - \lambda)^n v = 0\}$ of f_1 with eigenvalue λ are invariant under the actions v_n of any $v \in W$ and $n \in \mathbb{Z}$, since f_1 commutes with all v_n for $v \in W$. Since f_1 acts on every finite dimensional homogeneous space V_n and all eigenvalues of f_1 on $\mathbb{C}V$ are in the field K, V is a direct sum of generalized eigenspaces T_λ of f_1. Since e_1 acts on V_λ diagonally, V_λ is a W-module and so it is a direct sum of irreducible W-modules.

We will next show that $T_\lambda = V_\lambda$ for all λ. Suppose this is false and choose λ such that $V_\lambda \neq T_\lambda$. Let n be the minimal degree of T_λ / V_λ. As we explained as above, f_1 and v_n act on T_λ/V_λ for $v \in W$ and $n \in \mathbb{Z}$. Let X/V_λ be the eigenspace of f_1 in T_λ/V_λ. We note $X_n \neq (V_\lambda)_n$ and X/V_λ is a W-module by the same arguments as above, where P_n denotes $V_n \cap P$ for $P \subseteq V$. Hence there is an irreducible W-submodule B/V_λ of X/V_λ

satisfying $B_n \neq (V_\lambda)_n$. We note that since e_1 acts on X/V_λ as $\omega_1 - \lambda$, the highest weight of e_1 on X/V_λ is $n - \lambda$.

Let S be the submodule of V_λ generated by all irreducible W-submodules which are not isomorphic to B/V_λ. Set $(B/S)_n = (B_n + S)/S$. Since S is a direct factor of V_λ, $(f_1 - \lambda)(B/S)_n \neq 0$ and $(B/S)_n$ is the top module of B/S. We next show that the Zhu-algebra $A(W)$ of W acts on $(B/S)_n$.

In order to prove the above assertion, we have to explain an idea for the Zhu-algebra in [Zh]. We will treat a general case for a while. Let $A(W) = W/O(W)$ be the Zhu-algebra of W. For $v \in W$, $o(v)$ denotes the grade preserving operator of v. For a homogeneous element $v \in W_m$, if $Y^M(v, z) = \sum v_i z^{-i-1}$ is a vertex operator of v for a W-module M, then $o(v) = v_{m-1}$. It actually depends on the module M, but we write $o(v) = v_{m-1}$ formally.

Let $\tilde{R}$ be the ring generated by all v_i of $v \in W$ and $i \in \mathbb{Z}$ and let R be the subring generated by elements $\sigma = v^r_{i_r} \cdots v^1_{i_1}$ such that σ preserves the degree. Let I be the subspace of R generated by the elements $\sigma = v^r_{i_r} \cdots v^1_{i_1} \in R$ satisfying the following condition:

(C) There is an integer s such that $v^s_{i_s} \cdots v^1_{i_1}$ decreases the grade.

We note that if $M = \bigoplus_{i=0}^{\infty} M_i$ is a W-module, then $\sigma \in I$ acts on the top module M_0 trivially since $\sigma u = v^r_{i_r} \cdots (v^s_{i_s} \cdots v^1_{i_1} u) = 0$ for $u \in M_0$. Hence, R/I acts on the top module M_0. We admit an infinite sum of such elements if it is well defined in R. Clearly, I is a two-sided ideal of R and it is known that $A(W) = R/I$; see [Zh].

Let us go back to the proof. Since W is rational, $A(W)$ is a semi-simple. Since I acts on $(B/S)_n$ trivially, $A(W)$ acts on $(B/S)_n$. Since $A(W)$ is semi-simple, $(B/S)_n$ is a direct sum of irreducible $A(W)$-modules. By the choice of B and S, $(B/S)_n$ is a homogeneous $A(W)$-module. Since e is in the center of $A(W)$ by [Zh], e_1 acts on $(B/S)_n$ as a scalar times and so does f_1, which contradicts the facts that $(f_1 - \lambda)(B/S)_n \neq 0$. This completes the proof of Theorem 6.1. ∎

APPENDIX

Fusion Rules of Irreducible $L(\frac{4}{5}, 0)$-Modules

For $L(\frac{4}{5}, 0)$-modules, the following fusion rules are known (see [W]). In the following table, the numbers h denote $L(\frac{4}{5}, h)$ and $h_1\colon \cdots \colon h_t$ denotes $L(\frac{4}{5}, h_1) + \cdots + h(\frac{4}{5}, h_t)$.

Table C

0	$\frac{2}{5}$	$\frac{1}{40}$	$\frac{7}{5}$	$\frac{21}{40}$	$\frac{1}{15}$	3	$\frac{13}{8}$	$\frac{2}{3}$	$\frac{1}{8}$
$\frac{2}{5}$	$0:\frac{7}{5}$	$\frac{1}{8}:\frac{21}{40}$	$\frac{2}{5}:3$	$\frac{1}{40}:\frac{13}{8}$	$\frac{1}{15}:\frac{2}{3}$	$\frac{7}{5}$	$\frac{21}{40}$	$\frac{1}{15}$	$\frac{1}{40}$
$\frac{1}{40}$	$\frac{1}{8}:\frac{21}{40}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}$	$\frac{1}{40}:\frac{13}{8}$	$\frac{2}{5}:3:\frac{1}{15}:\frac{2}{3}$	$\frac{1}{40}:\frac{13}{8}:\frac{21}{40}:\frac{1}{8}$	$\frac{21}{40}$	$\frac{7}{5}:\frac{1}{15}$	$\frac{21}{40}:\frac{1}{40}$	$\frac{1}{15}:\frac{2}{5}$
$\frac{7}{5}$	$\frac{2}{5}:3$	$\frac{1}{40}:\frac{13}{8}$	$0:\frac{7}{5}$	$\frac{1}{8}:\frac{21}{40}$	$\frac{2}{3}:\frac{1}{15}$	$\frac{2}{5}$	$\frac{1}{40}$	$\frac{1}{15}$	$\frac{21}{40}$
$\frac{21}{40}$	$\frac{1}{40}:\frac{13}{8}$	$\frac{2}{5}:3:\frac{1}{15}:\frac{2}{3}$	$\frac{1}{8}:\frac{21}{40}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}$	$\frac{1}{8}:\frac{21}{40}:\frac{13}{8}:\frac{1}{40}$	$\frac{1}{40}$	$\frac{2}{5}:\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{1}{15}:\frac{7}{5}$
$\frac{1}{15}$	$\frac{1}{15}:\frac{2}{3}$	$\frac{1}{40}:\frac{13}{8}:\frac{21}{40}:\frac{1}{8}$	$\frac{2}{3}:\frac{1}{15}$	$\frac{1}{8}:\frac{21}{40}:\frac{13}{8}:\frac{1}{40}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}:3:\frac{2}{5}$	$\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$	$\frac{1}{40}\quad\frac{21}{40}$
3	$\frac{7}{5}$	$\frac{21}{40}$	$\frac{2}{5}$	$\frac{1}{40}$	$\frac{1}{15}$	0	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{13}{8}$
$\frac{13}{8}$	$\frac{21}{40}$	$\frac{7}{5}:\frac{1}{15}$	$\frac{1}{40}$	$\frac{2}{5}:\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{1}{8}$	$0:\frac{2}{3}$	$\frac{1}{8}:\frac{13}{8}$	$\frac{2}{3}:3$
$\frac{2}{3}$	$\frac{1}{15}$	$\frac{21}{40}:\frac{1}{40}$	$\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$	$\frac{2}{3}$	$\frac{1}{8}:\frac{13}{8}$	$0:\frac{2}{3}:3$	$\frac{1}{8}:\frac{13}{8}$
$\frac{1}{8}$	$\frac{1}{40}$	$\frac{1}{15}:\frac{2}{5}$	$\frac{21}{40}$	$\frac{1}{15}:\frac{7}{5}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{13}{8}$	$\frac{2}{3}:3$	$\frac{1}{8}:\frac{13}{8}$	$0:\frac{2}{3}$

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